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**Indeterminacy of effect components of factorial experiments arising from the ways of expression**

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FORMULAS: PART **0**

**(A1)** $y = X.cc + vv = yy + vv$

provided that $yy = X.cc$

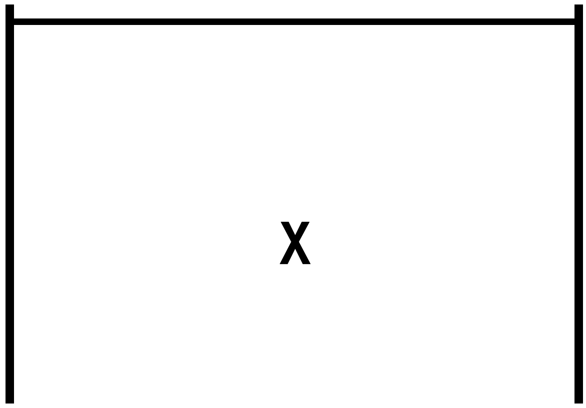
**(A2)** $y = X.cv + vy = yv + vy$

provided that $yv = X.cv$

**(A3)** $X'X.cv = X'.y$ so

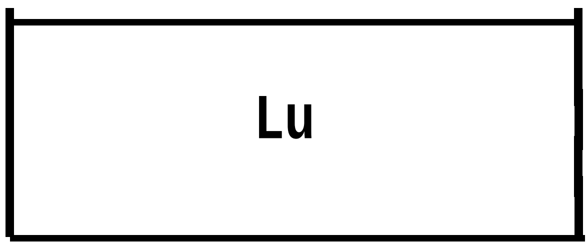
a solution $cv = (X'X)^{-}.X'.y$

PART **1**




**(A4)**

$Lu = K.X$




**(A5)**

$J.Lu = X$

## PART 2

**(A6)**

$$Lu^{-}o = Lu\_1.Lu^{-}1 + Lu\_2.Lu^{-}2 + \ldots + Lu\_qu.Lu^{-}qu$$

**(A7)**

$$Lu^{-}o.cc$$
$$Lu^{-}o.cv$$

**(A′1)** $y = Xp.ccP + vvP = yyP + vvP$

provided that $yyP = Xp.ccP$

**(A′2)** $y = Xp.cvP + vyP = yvP + vyP$

provided that $yvP = Xp.cvP$

**(A′3)** $Xp'Xp.cvP = Xp'.y$ so

a solution $cvP = (Xp'Xp)^{-}.Xp'.y$

## PART 3

**(A8)** $SS\_yv = yv'.yv = cv'.X'X.cv$

$$= (Lu.cv)'.J'J.(Lu.cv)$$

**(A′8)** $SS\_yvP = yvP'.yvP = cvP'.Xp'Xp.cvP$

$$= (L.cvP)'.J'J.(L.cvP)$$

# PART A

**(A9) SS_yv = yv′.yv = cv′.X′X.cv**
**= (Lu.cv_1)′.J′J.(Lu.cv_1)**
**= (Lu.cv_2)′.J′J.(Lu.cv_2)** provided that **cv_2 = cv_1 + cf** and **Lu.cf = 0**

**(A′9) SS_yvP = yvP′.yvP = cvP′.Xp′Xp.cvP**
**= (L.cvP_1)′.J′J.(L.cvP_1)**
**= (L.cvP_2)′.J′J.(L.cvP_2)** provided that **cvP_2 = cvP_1 + cfP** and **L.cfP = 0**

# PART B

***（vy:= y-yv） *（vyP:= y-yvP）**

**(A10) SS_y = y′.y = SS_yv + SS_vy**
**(A11) SS_yv = yv′.yv （yv = X.cv）**
**= (Lu.cv)′.J′J.(Lu.cv) （X′X.cv = X′.y）**
**(A12) SS_vy = vy′.vy （vy.yv = 0）***
**(A13) SS_yv = SS_y - SS_vy = R(cv)**

**(A′10) SS_y = y′.y = SS_yvP + SS_vyP**
**(A′11) SS_yvP = yvP′.yvP （yvP = Xp.cvP）**
**=(L.cvP)′.J′J.(L.cvP) （Xp′Xp.cvP = Xp′.y）**
**(A′12) SS_vyP = vyP′.vyP （vyP.yvP = 0）***
**(A′13) SS_yvP = SS_y - SS_vyP = R(cvP)**

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**Indeterminacy of effect components of factorial experiments arising from the ways of expression**

Tadao SHIBAYAMA (Retired: Nagoya Municipal Industrial Research Institute)

PART **0 –** The discussion is to be started from the structure equation (A1) of a model and the observation equation (A2), both on the basis of a design matrix `X`. The true effect elements column vector `cc` in eqn (A1) is represented usually by a greek small letter (beta) and the error column vector `vv` by another greek small letter (epsilon). The fitted effect elements column vector `cv` in eqn (A2) is represented usually by a roman small letter (b) and the residue column vector `vy` by another roman small letter (e). The normal equation (A3) follows from least sum of squared residues but with some indeterminacy in the solution.

**1 –** The design matrix `X` is reduced to a full estimable matrix `Lu` by operation of a contraction matrix `K` as shown in eqn (A4) removing plural rows of the design matrix and extracting all linearly independent rows from the design matrix. The design matrix `X` is restored from the full estimable matrix `Lu` by operation of a restoration matrix `J` as shown in eqn (A5).

**2 –** Rows `Lu¯1`, `Lu¯2`, ⋯, `Lu¯qu` of a full estimable matrix `Lu`, `qu` rows (, say,) in total, may be linearly combined to generate an estimable row vector `Lu¯o` by operation of a set of arbitrary coefficients `Lu_1`, `Lu_2`, ⋯, `Lu_qu` of combination as shown in eqn (A6). The estimable row vector `Lu¯o` is multiplied to a true or fitted effect elements column vector (`cc` or `cv`) as shown in expn (A7), it generates a true or fitted estimable function `Lu¯o.cc` or `Lu¯o.cv`.

If some rows of a full estimable matrix `Lu` are replaced arbitrarily by a row of zeroes, one by one, to form a part estimable matrix `L`, and the restoration matrix `J` is operated, then, a part design matrix `Xp` is generated. Even with such a part design matrix `Xp`, and with an observed response column vector `y` as given, a structure equation (A'1), an observation equation (A'2) and the normal equation (A'3) of least sum of squared residues may be established.

**3 –** Sum of squares `SS_yv` or `SS_yvP` of the fitted response `yv` (`:= X.cv`) or `yvP` (`:= Xp.cvP`), respectively, may be calculated as shown in eqn (A8) or (A'8), respectively.

**A –** Even with the indeterminacy of the solution of the normal equation, the sum of squares of the fitted response, `SS_yv` or `SS_yvP`, as calculated on the basis of the full or part estimable matrix `Lu` or `L`, respectively, is defined uniquely as it proves as shown in eqn (A9) or (A'9).

**B –** Because of the procedure of the least sum of squared residues, the fitted response column vector `yv` or `yvP` is orthogonal to the fitted residue column vector `vy` or `vyP`, respectively.

Consequently, the sum of squares of the observed response, `SS_y`, is decomposed to a sum of the sum of squares of the fitted response (`SS_yv` or `SS_yvP`) and the sum of squares of the fitted residue (`SS_vy` or `SS_vyP`) as shown in eqns (A10)-(A12) or (A'10)-(A'12).

The term `X.cv (=yv)` or `Xp.cvP (=yvP)` of the observation equation is fitted to the observed response `y` in a way as shown in the parentheses appended to eqns (A11) and (A12), or to eqns (A'11) and (A'12). The sum of squares of observed response, `SS_y`, is reduced to the sum of squares of fitted residue, `SS_vy` or `SS_vyP`. The reduction `R(cv)` or `R(cvP)` is equal to the sum of squares of fitted response, `SS_yv` or `SS_yvP`, respectively, as defined in eqns (A13) and (A'13). That is to be compared to the sum of squares of fitted residue, `SS_vy` or `SS_vyP`, respectively, in order to test if that is significantly larger than the fitted residue.

Columns of the restoration matrix `J` are to be 'exactly' proved to be linearly independent to each other, provided that all rows of the full estimable matrix `Lu` are linearly independent to each other. By a transformation, columns of the restoration matrix `J` are orthogonalized to each other with some change in rows of the full estimable matrix `Lu`, so that the columns to be some of the base column vectors in response space. A multidimensional normal distribution may be postulated as the basis of the null hypothesis for such the significance test above.

An example of hypothesis testing by SAS/STAT$^{R}$ GLM procedure.
Cf. SAS/STAT$^{R}$ User's Guide, version 6, Volume 2, Chap. 24, p. 932-936,
and Volume 1, Chap. 9, p. 109-110, and around as it may concern.

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The structure equation of a model. `y = yy + vv = X.cc + vv`

The response column vector, with the element for the treatment-run `AaBb.r` of the `r`-th run of the level `a` of the factor `A` and the level `b` of the factor `B` such as follows

`A1B1.1  A1B1.2  A1B2  A2B1  A2B2.1  A2B2.2  A3B1.1  A3B1.2  A3B2.1  A3B2.2 :`

- as observed `y = ( y1,  y2,  y3,  y4,  y5,  y6,  y7,  y8,  y9,  y10)'`
- of the true values `yy = (yy1, yy2, yy3, yy4, yy5, yy6, yy7, yy8, yy9, yy10)'`

The error column vector
of the true samples `vv = (vv1, vv2, vv3, vv4, vv5, vv6, vv7, vv8, vv9, vv10)'`

The effect elements column vector, of the true values, `cc := (ccM, ccA1, ccA2, ccA3, ccB1, ccB2, ccAB11, ccAB12, ccAB21, ccAB22, ccAB31, ccAB32)'`

The design matrix.

$$
\begin{array}{rl}
 & \begin{array}{cccccccccccc} \mathtt{ccM} & \multicolumn{3}{c}{\mathtt{ccA1,2,3}} & \multicolumn{2}{c}{\mathtt{ccB1,2}} & \multicolumn{6}{c}{\mathtt{ccAB11,\ 12,\ 21,\ 22,\ 31,\ 32}} \end{array} \\
\mathtt{X} = \begin{array}{r} \mathtt{A1B1.1}\\ \mathtt{A1B1.2}\\ \mathtt{A1B2}\\ \mathtt{A2B1}\\ \mathtt{A2B2.1}\\ \mathtt{A2B2.2}\\ \mathtt{A3B1.1}\\ \mathtt{A3B1.2}\\ \mathtt{A3B2.1}\\ \mathtt{A3B2.2} \end{array} &
\begin{bmatrix}
1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0\\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0\\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0\\
1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1\\
1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\end{array}
$$

The arid design matrix (`X*` of p.110, The Volume 1), with the duplicate rows deleted:

$$
\mathtt{XX} = \begin{bmatrix}
1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0\\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0\\
1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{array}{l} 1\\ 2\\ 3\\ 4\\ 5\\ 6 \end{array}
$$

The full estimable matrix, with the linearly independent rows arbitrarily extracted:

$$
\mathtt{Lu} = \begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1\\
0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1\\
0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & -1\\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & -1\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1
\end{bmatrix}
\begin{array}{l} 6\\ 2-6\\ 4-6\\ 5-6\\ 1-2-5+6\\ 3-4-5+6 \end{array}
$$

******************************************************************************

$$
\mathtt{J} = \begin{bmatrix}
-1 & 1 & 0 & -1 & 1 & 0\\
-1 & 1 & 0 & -1 & 1 & 0\\
-1 & 1 & 0 & 0 & 0 & 0\\
1 & 0 & 1 & 1 & 0 & 1\\
1 & 0 & 1 & 0 & 0 & 0\\
1 & 0 & 1 & 0 & 0 & 0\\
1 & 0 & 0 & 1 & 0 & 0\\
1 & 0 & 0 & 1 & 0 & 0\\
1 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

The restoration matrix.

`X = J.Lu`

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A full estimable matrix, `LuI`, `LuII`, `LuIII` or `Lu` (above), each in SAS/STAT$^{R}$ User's Guide, version 6, Volume 2, p. 932-936, **Output 24.3-6**, as of Type I, II, III, or General, respectively, and the sweep operator (`LL`'s) such that `LuI=LLI.Lu`, `LuII=LLII.Lu` or `LuIII=LLIII.Lu`.

********************************************************************************

$$\mathrm{LuI} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & 1/6 & -1/6 & 2/3 & 1/3 & 0 & 0 & -1/2 & -1/2 \\ 0 & 0 & 1 & -1 & -1/6 & 1/6 & 0 & 0 & 1/3 & 2/3 & -1/2 & -1/2 \\ 0 & 0 & 0 & 0 & 1 & -1 & 2/7 & -2/7 & 2/7 & -2/7 & 3/7 & -3/7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix}$$

$$\mathrm{LuII} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 & 0 & 13/21 & 8/21 & -1/21 & 1/21 & -4/7 & -3/7 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1/21 & -1/21 & 8/21 & 13/21 & -3/7 & -4/7 \\ 0 & 0 & 0 & 0 & 1 & -1 & 2/7 & -2/7 & 2/7 & -2/7 & 3/7 & -3/7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix}$$

$$\mathrm{LuIII} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 & 0 & 1/2 & 1/2 & 0 & 0 & -1/2 & -1/2 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1/2 & 1/2 & -1/2 & -1/2 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1/3 & -1/3 & 1/3 & -1/3 & 1/3 & -1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix}$$

********************************************************************************

$$\mathrm{LLI} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1/6 & 2/3 & 0 \\ 0 & 0 & 1 & -1/6 & 0 & 1/3 \\ 0 & 0 & 0 & 1 & 2/7 & 2/7 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{LLII} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 13/21 & -1/21 \\ 0 & 0 & 1 & 0 & 1/21 & 8/21 \\ 0 & 0 & 0 & 1 & 2/7 & 2/7 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{LLIII} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 1/3 & 1/3 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

********************************************************************************

**-1.** The sweep operators should be derived on the basis of descriptions of Chap. 9, Volume 1, p.109-124, though explicit and complete descriptions of the steps are often not so popular.

**-2.** Construction of rows of any particular full or part estimable matrix (`Lu` or `L`) for estimation of Type I, II, III etc., respectively, from that of Type General is important. It is, perhaps, done by part estimable matrices (`L`) built of the full matrix (`Lu`) with rows zeroed. And, if necessary, done by 'curt' estimable matrices (`Lv`), too, built of the full matrix (`Lu`) with columns zeroed.

**-3.** Many estimable functions are formed arbitrarily (by linear transformation or so) of rows of the full estimable matrix (`Lu`) yet all governed by the linear independence and the estimability. It secures general applicability of the scheme but with complication by the indeterminacy of the effect elements, to be relieved perhaps by the usual constraints on the basis of isolability.

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Restoration matrix, JI, JII or JIII, for the full estimable matrix (above), LuI, LuII or LuIII, of Type I, II or III, respectively, such as built of the restoration matrix J of the General type by the inverse sweep operator (KK's) such that JI=J.KKI, JII=J.KKII or JIII=J.KKIII.

*****************************************************************************

$$JI = \begin{bmatrix} -1 & 1 & 0 & -7/6 & 2/21 & 3/7 \\ -1 & 1 & 0 & -7/6 & 2/21 & 3/7 \\ -1 & 1 & 0 & -1/6 & -13/21 & 1/21 \\ 1 & 0 & 1 & 1/6 & -1/3 & 1/3 \\ 1 & 0 & 1 & 1/6 & -13/21 & -8/21 \\ 1 & 0 & 1 & 1/6 & -13/21 & -8/21 \\ 1 & 0 & 0 & 1 & 2/7 & -2/7 \\ 1 & 0 & 0 & 1 & 2/7 & -2/7 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The restoration matrix.

X = JI.LLI

$$JII = \begin{bmatrix} -1 & 1 & 0 & -1 & 2/3 & 0 \\ -1 & 1 & 0 & -1 & 2/3 & 1/3 \\ -1 & 1 & 0 & 0 & -13/21 & 1/21 \\ 1 & 0 & 1 & 1 & -1/3 & 1/3 \\ 1 & 0 & 1 & 0 & -1/21 & -8/21 \\ 1 & 0 & 1 & 0 & -1/21 & -8/21 \\ 1 & 0 & 0 & 1 & -2/7 & -2/7 \\ 1 & 0 & 0 & 1 & -2/7 & -2/7 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The restoration matrix.

X = JII.LLII

$$JIII = \begin{bmatrix} -1 & 1 & 0 & -1 & 5/6 & 0 \\ -1 & 1 & 0 & -1 & 5/6 & 0 \\ -1 & 1 & 0 & 0 & -1/2 & 0 \\ 1 & 0 & 1 & 1 & -1/3 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1/6 \\ 1 & 0 & 1 & 0 & 0 & 1/6 \\ 1 & 0 & 0 & 1 & -1/3 & -1/2 \\ 1 & 0 & 0 & 1 & -1/3 & -1/2 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The restoration matrix.

X = JIII.LLIII

*****************************************************************************

$$KKI = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1/6 & -13/21 & 1/21 \\ 0 & 0 & 1 & 1/6 & -1/21 & -8/21 \\ 0 & 0 & 0 & 1 & -2/7 & -2/7 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$KKII = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -13/21 & 1/21 \\ 0 & 0 & 1 & 0 & -1/21 & -8/21 \\ 0 & 0 & 0 & 1 & -2/7 & -2/7 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$KKIII = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1/2 \\ 0 & 0 & 0 & 1 & -1/3 & -1/3 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

*****************************************************************************

PART **4** **Cf.** PART **3**

# Another expression of eqn (A8) or (A'8)

$$SS\_yv = yv'.yv = cv'.X'X.cv$$
$$= (Lu.cv)'.J'J.(Lu.cv) \quad (A8)$$

(A14) $X'X.cv = X'.y = (J.Lu)'.y$ :

$cv = (X'X)^{-}.(J.Lu)'.y = (X'X)^{-}Lu'.J'y$

(A15) $Lu.cv = Lu(X'X)^{-}Lu'.J'y$

(A16) $J'y = (Lu(X'X)^{-}Lu')^{-1}.(Lu.cv)$

(A17) $(Lu.cv)'.J'y = (J.Lu.cv)'.y$
$= (Lu.cv)'.(Lu(X'X)^{-}Lu')^{-1}.(Lu.cv)$
$= (X.cv)'.y = cv'.X'y = cv'.(X'X).cv$
$= (X.cv)'(X.cv) = yv'.yv = SS\_yv$

(A18) $SS\_yv$
$= (Lu.cv)'.[Lu(X'X)^{-}Lu']^{-1}.(Lu.cv)$

(A'18) $SS\_yvP$ **similarly with** eqn (A'8)
$= (L.cvP)'.[L(Xp'Xp)^{-}L']^{-1}.(L.cvP)$

*******************************************************************************

**-4.** The equation (A18) or (A'18) is identical with that at the end of the section **ESTIMABILITY** in Chap. 9, Volume 1, p.109-110, the last line of the section just around the middle of p.110.

**-5.** The matrix $[Lu(X'X)^{-}Lu]^{-1}$ in eqn (A18) or $[L(Xp'Xp)^{-}L']^{-1}$ in eqn (A'18) may be replaced consistently by the matrix J'J in eqn (A11) or (A'11) in PART **B** as shown.

**-6.** The full estimable matrix LuI, LuII or LuIII and the sweep operator LLI, LLII or LLIII reflect the procedure of solution of the normal equation by Type I, II or III estimation, respectively. A'full'estimable matrix may be linearly transformed to another, so an estimable function Lu¯o.cvP (cf. expn (A7)) may be generated in a 'part' estimable matrix arbitrarily.

**-7.** Each sum of squares SS_yvP shall be tesed of a null hypothesis. If necessary, columns of the restoration matrix J should be orthogonalized to each other by an inverse sweep operator KK, with rows of the estimable matrix transformed by a sweep operator LL ($=KK^{-1}$). Such the test might be an approximation only, without the orthogonality of the columns.

**-8.** Often the effect components are too much flexible under the conception of estimability and reduction. A more specific formulation may be preferable and possible, sometimes.