Methodologies for calculating
Confidence Interval of Loss Given Default

- Probability of default (PD), Loss given default (LGD),Exposure at Default(EAD) are the three key parameters in the calculation of the minimum regulatory capital requirements under the Basel internal ratings-based framework.
- We would like to calculate the confidence interval of those parameters.

- The confidence interval of PD can be derived from the approximation below.

- The event “the client default or not” follows a Bernoulli distribution.
- The number of defaults in n Bernoulli trials follows binomial distribution. If the number of trials are big enough, the binomial distribution can be approximated to normal distribution.

- Is it possible to get confidence interval of LGD in the similar way?
- Roughly speaking,

  LGD=1-RecoveryRate

  =LossAmount/OutstandingAt Default

Every dollar lent out can be a loss. The loss amount can be regarded as bad dollars from n Bernoulli trials. And the confidence interval of LGD can use the same PD approach.

This approach distinguishes every dollar to be a trial.

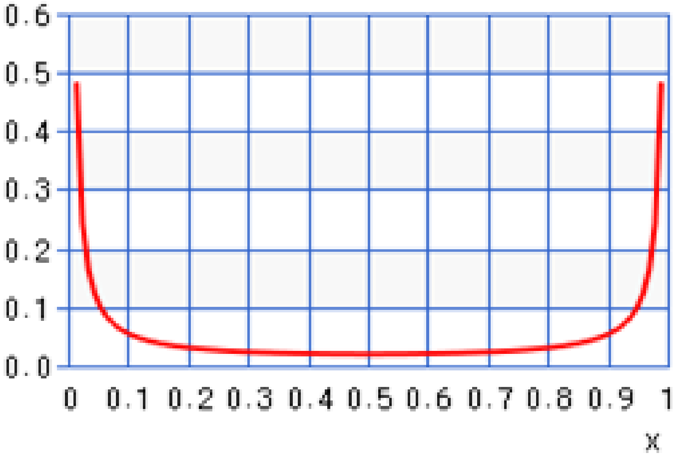
Or using binary transformation approach to transform each LGD observation into good dollars and bad dollars.

For example: An account with an LGD of 25% can be seen as 75 good cases and 25 bad cases.

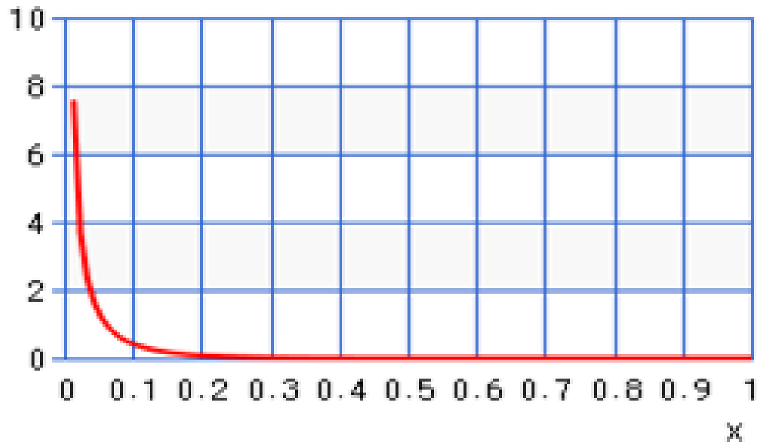
This approach amplifies the original data size to 100 times.

- Both approaches start from loss dollars instead of LGD itself.
- The LGD itself has an unusual distribution, which can be Bi-modal with modes in end-point values 0 and 1 or be Uni-modal distributions with mode in either 0 or 1.

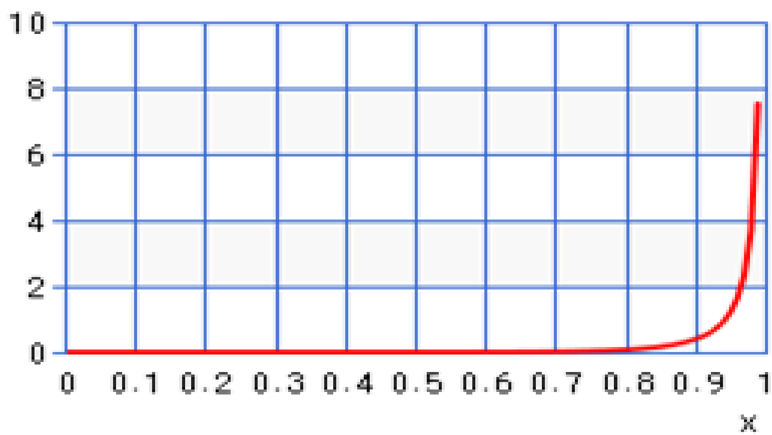
- Bi-modal distribution



Uni-modal distribution with mode in 0



Uni-modal distribution with mode in 1



Can we get the confidence interval started from the beta distribution?

Standard Beta distribution is a continuous distribution with probability density function f given by

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$$

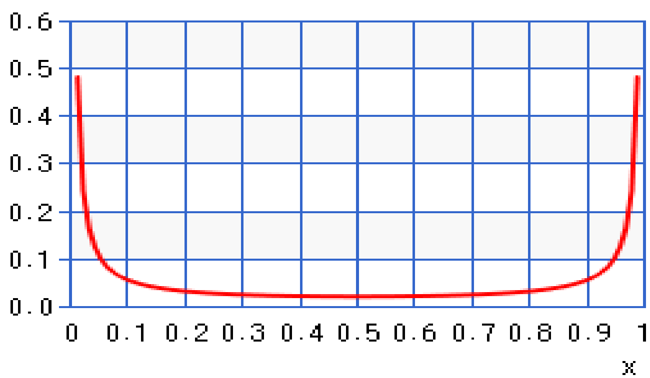
where 0<x<1

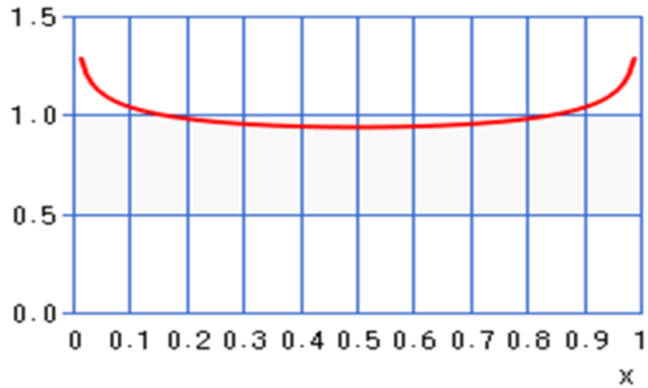
The mean and variance of X are:

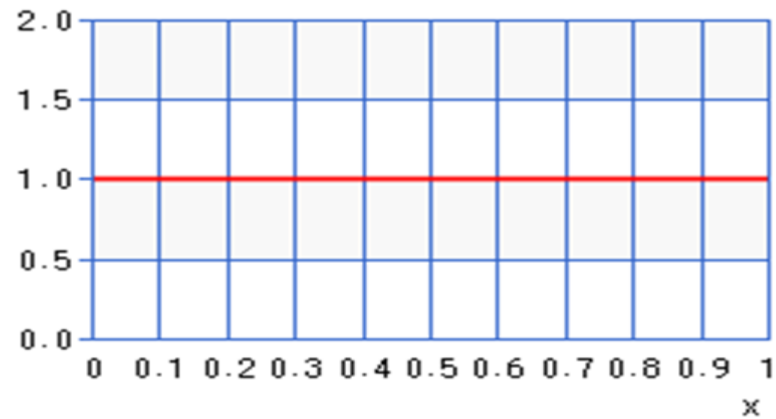
$$E(x) = \frac{a}{a+b}$$

$$Var(x) = \frac{ab}{(a+b)^2(a+b+1)}$$

- Below are the probability density function plot of beta distribution with a and b changes.






2.0
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1.0
0.5
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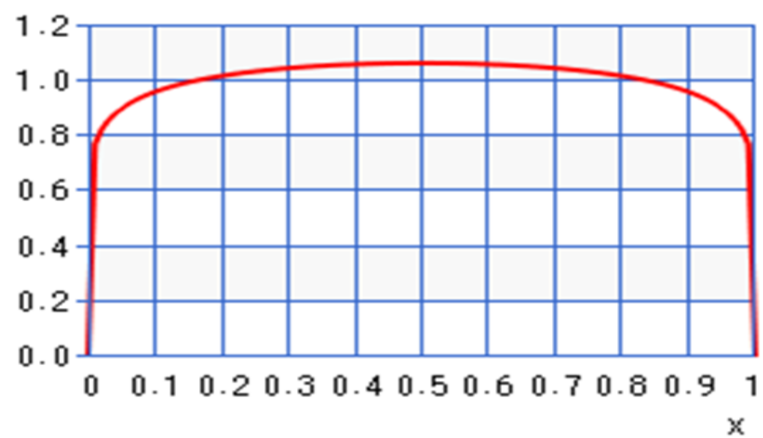
0.8
0.9
1
x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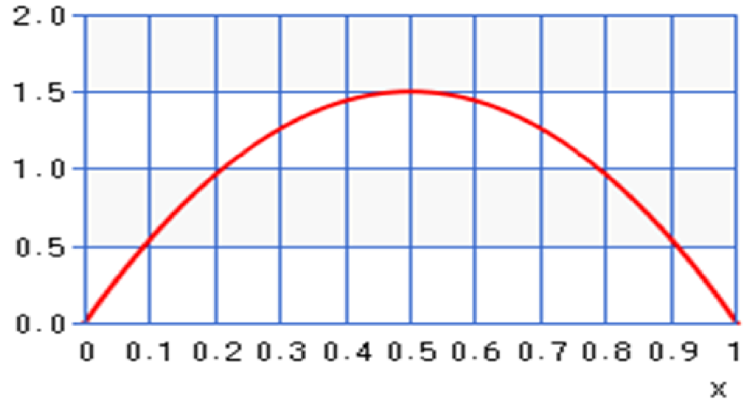

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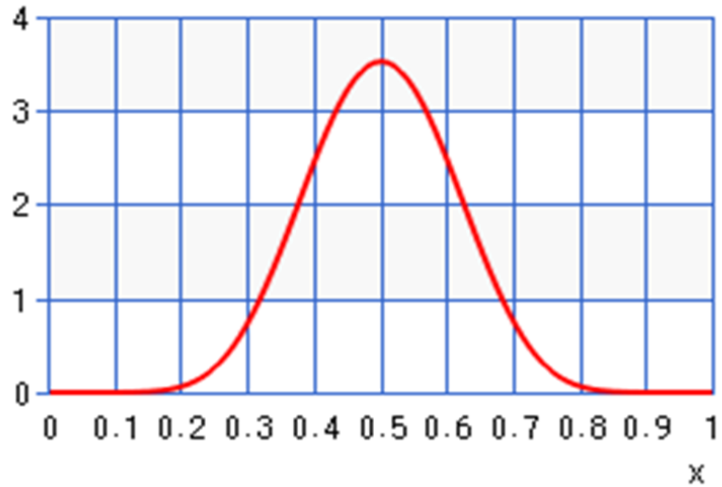

4
3
2
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0
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
x

- If a and b are approximately equal and large enough, Beta distribution is approximately normal. Then confidence interval is easy to get.
- What if a and b are not equal or both of them are small ?

Normalize LGD

$$NLGD = \Phi^{-1}[F(LGD, \alpha, \beta)]$$

where

$$\alpha = (E^2(LGD) * (1 - E(LGD)) / Var(LGD) - E(LGD)$$

$$\beta = \alpha * (1 / E(LGD) - 1)$$

we assume NLGD1, NLGD2,…, NLGDn to be a random sample drawn from a population with $NLGD_i \sim N(\mu, \sigma^2)$.

The confidence interval for μ is:

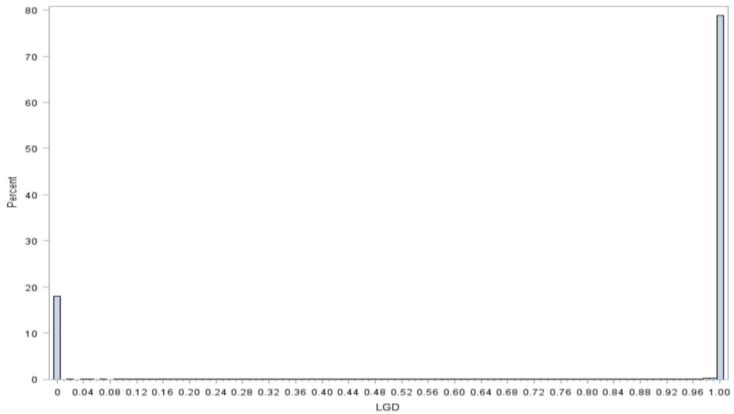
$$\overline{X} \pm m \frac{S}{\sqrt{n}}$$

Convert LGD observations from [0, 1] to (0, 1)

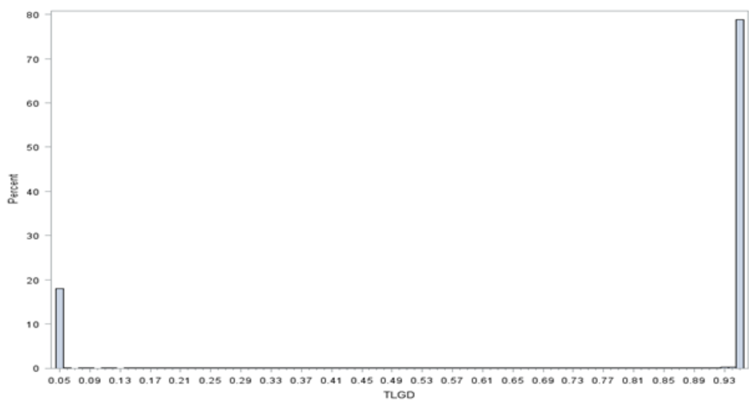
- For some reason, the LGD might be bigger than 1.
- To match the required boundary values of the standard beta distribution variable, some conversion needed here.
- There are many ways to do it. For example, we convert the original LGD to:

$$TLGD = \frac{LGD + 0.05}{\max + 0.05} * 0.95$$

- Distribution of original LGD(bounded at 0 and 1)



- Distribution of converted LGD, denoted by TLGD(bounded within 0 and 1)



Normalize converted LGD

$$NLGD = \Phi^{-1}[F(TLGD, \alpha, \beta)]$$

where

$$\alpha = (E^2(TLGD) * (1 - E(TLGD)) / Var(TLGD) - E(TLGD)$$

$$\beta = \alpha * (1 / E(TLGD) - 1)$$

- We transform NLGD confidence interval boundary values back to TLGD to get the corresponding TLGD lower and upper bounds by

- And then convert back to original LGD:

$$TLGD = F^{-1}(\Phi(NLGD), \alpha, \beta)$$

$$LGD = TLGD(\max + 0.5) - 0.5$$

Approach 2:Asymptotic normality of MLE

Concept 1: Maximum Likelihood Estimator

Consider X1, X2,…,Xn to be an iid sample drawn from a population with probability density function $f(x_i;\theta)$, where θ is a (k * 1)vector of parameters that characterize $f(x_i;\theta)$.

The likelihood function of the sample is the joint PDF

$$L(\theta) = f(x_1,\ldots,x_n;\theta) = \prod_{i=1}^{n} f(x_i;\theta)$$

The maximum likelihood estimator of θ, denoted by $\hat{\theta}_{mle}$, maximizes L(θ):

$$L(\hat{\theta}_{mle}) \geq L(\theta), \forall \theta$$

Property of MLE:

Let $X_1, \ldots, X_n$ be a sample of size n from a distribution for which the pdf is $f(x \mid \theta)$, with θ the unknown parameter. Assume that the true value of θ is θ_0, and the MLE of θ is $\hat{\theta}$. Then the probability distribution of $\sqrt{n\mathrm{I}(\theta_0)}(\hat{\theta} - \theta_0)$ tends to a standard normal distribution.

Where $\mathrm{I}(\theta_0)$ is the fisher information, which can be interpreted as the "information about the parameters contained in the observation x."

Concept 2: Fisher Information

Let X be a random variable with p.d.f. $f(x;\theta), \theta \in \Omega$, where Ω is the parameter space of θ. Fisher information, denoted by $\mathrm{I}(\theta)$ is defined as:

$$\mathrm{I}(\theta) = \int_{-\infty}^{\infty} [\frac{\partial \ln f(x;\theta)}{\partial \theta}]^2 f(x;\theta) dx \text{ or } \mathrm{I}(\theta) = -\int_{-\infty}^{\infty} \frac{\partial^2 \ln f(x;\theta)}{\partial \theta^2} f(x;\theta) dx$$

Further, suppose that X1 X2… Xn is a random sample drawn from a distribution that has PDF $f(x_i;\theta)$. Thus the Fisher information in the random sample is n times the Fisher information in one observation. That is:

$$\mathrm{I}_n(\theta) = n\mathrm{I}(\theta)$$

- Continuing from the defined NLGD above, we assume NLGD1, NLGD2,…, NLGDn is an independent and identical distributed sample drawn from a population with normal distribution with mean μ and variance σ^2.

- According to the asymptotic distribution of MLE property described above, we have

$$\sqrt{n\mathrm{I}(\mu)}(\hat{\mu}-\mu) \sim N(0,1)$$

- An approximate confidence interval for μ is:

$$\hat{\mu}_{mle} \pm z_{\alpha}\sqrt{\frac{1}{n\mathrm{I}(\mu)}}$$

Where

- $\hat{\mu}_{mle}$ is the maximum likelihood estimator of μ;
- $\mathrm{I}(\mu)$ is fisher information of μ;
- $z_{(\alpha)}$ is the desired confidence level quantile of the standard normal distribution;
- n is the sample size.

The Fisher information of μ is

$$\mathrm{I}(\mu) = \frac{1}{\sigma^2}$$

The maximum likelihood estimator for μ and σ^2 are:

$$\hat{\mu}_{mle} = \bar{x}$$

$$\hat{\sigma}^2{}_{mle} = \frac{1}{n}\sum_{i=1}^{n}(x_i - \hat{\mu}_{mle})^2$$

Where x_i represent $NLGD_i$.

The $\hat{\mu}_{mle}$ is unbiased whereas $\hat{\sigma}^2{}_{mle}$ is a biased estimator because

$$E(\hat{\sigma}^2{}_{mle}) = \frac{(n-1)\sigma^2}{n} \neq \sigma^2$$

We would like to find the unbiased estimator of σ^2. That is:

$$\hat{\sigma}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

- The Above two approaches are both based on normal distributed NLGD.
- Is it possible to work on original LGD directly?

- Recall the mean of the standard Beta distribution is:

$$\theta = \boldsymbol{a}/(\boldsymbol{a}+\boldsymbol{b})$$

According to the asymptotic properties of maximum likelihood estimator, the probability distribution of:

$$\sqrt{n\mathrm{I}(\theta_0)}(\hat{\theta}-\theta_0)$$

tends to be a standard normal distribution.

We need to calculate the maximum likelihood estimator of θ and its Fisher information from the beta distribution with the mean as one of the parameters.

Approach 3:Central Limit Theorem

Theorem: Let X1,X2,…,Xn denote the observations of a random sample from a distribution that has mean μ and positive variance σ^2. Then the random variable

$$Y_n = \frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n}\sigma} = \sqrt{n}(\bar{X}_n - \mu)/\sigma$$

has a limited distribution that is normal with mean zero and variance 1.

The theorem is interpreted as: when n is a large, fixed positive integer, the random variable $\bar{X}$ has an approximate normal distribution with mean μ and variance σ^2/n.

- The Central Limit Theorem is valid for any distribution with a large sample size.

In general, LGD is distinctly not normal distributed. We use the student t distribution to construct the confidence interval.

$$\bar{X} \pm t_{(\alpha, n-1)} \frac{S}{\sqrt{n}}$$

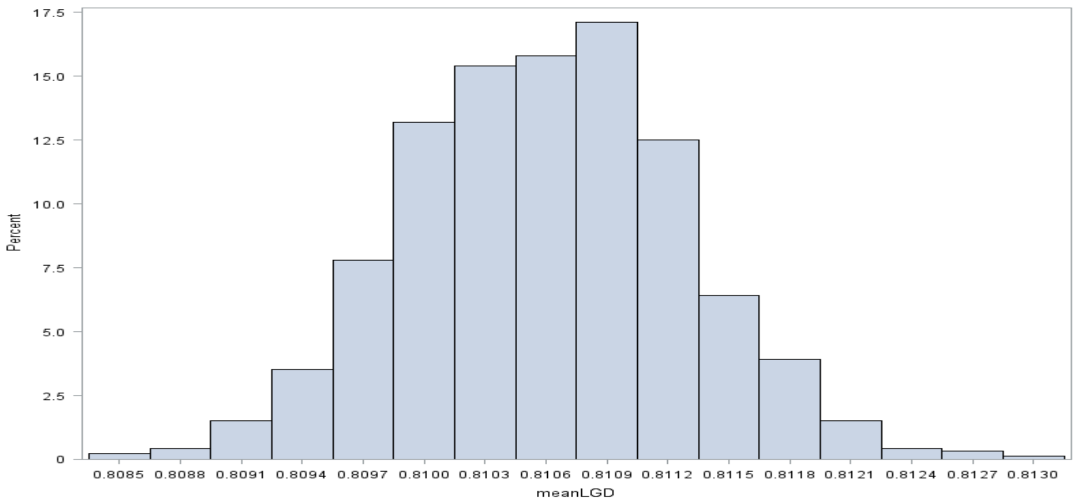
Approach 4: **Bootstrapping**

- The Bootstrapping procedure is distribution-independent. It provides an indirect method to assess the properties of the distribution underlying the sample and parameters of interest that are derived from this distribution.

- The simplest bootstrap method can be used to construct the confidence interval for the mean of the sampling distribution. For instance, we take the original data set of size N, and sampling from the dataset with replacement to form a new sample with same size N. We repeat this process a large number of times, and compute the mean for each bootstrapped sample. The histogram of bootstrap means is then constructed and the desired confidence interval can be found too.

- Below is an example.

Bootstrapped sampling mean distribution



- The bootstrapping distribution plot confirmed that the sampling mean distribution is bell shaped.

summary

- Above all, the confidence interval for unusual distributed variable can be done by binomial imitation, normalization, asymptotic normality of MLE, central limit theory and bootstrapping etc.
- Normalization might change the original data property too much and lead us far from the reality. Asymptotic normality of MLE always involve complicated calculation, sometime may challenge the software tools.
- Those approaches can also be applied on EAD confidence interval constructing.