Risk Aggregation and Economic Capital
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Introduction

The following paper discusses challenges faced by financial institutions in the areas of risk aggregation and economic capital. SAS has responded to these challenges by delivering an integrated risk offering, SAS® Risk Management for Banking. The solution meets the immediate requirements banks are looking for, while providing a framework to support future business needs.

Risk management for banks involves risk measurement and risk control at the individual risk level, including market risk for trading books, credit risk for trading and banking books, operational risks and aggregate risk management. In many banks, aggregate risk is defined using a rollup or risk aggregation model; capital, as well as capital allocation, is based on the aggregate risk model. The aggregate risk is the basis for defining a bank’s economic capital, and is used in value-based management such as risk-adjusted performance management.

In practice, different approaches to risk aggregation can be considered to be either one of two types: top-down or bottom-up aggregation. In the top-down aggregation, risk is measured on the sub-risk level such as market risk, credit risk and operational risk; subsequently, risk is aggregated and allocated using a model of risk aggregation.

In the bottom-up aggregation model, the sub-risk levels are aggregated bottom-up using a joint model of risk and correlations between the different sub-risks that drive risk factors. In this method the different risk factors for credit, market, operational risk, etc. are simulated jointly.

While bottom-up risk aggregation may be considered a preferred method for capturing the correlation between sub-risks, sometimes bottom-up risk aggregation is difficult to achieve because some risks are observed and measured at different time horizons. For example, trading risk is measured intraday or at least daily while operational risk is typically measured yearly. The difficulty in assigning a common time horizon for risks that are subject to integration does not necessarily become easier using a top-down method. Indeed, this method also requires sub-risks to be measured using a common time horizon. However, a distinctive difference between the approaches is that for the top-down method, one is usually concerned with specifying correlations on a broad basis between the different sub-risks (e.g., correlation between credit and market risk or correlation between operational risk and market risk). In contrast, the bottom-up approach requires the specification of all the underlying risk drivers of the sub-risks. Such detailed correlations may be natural to specify for some sub-risks such as trading market and credit risk, but not for others such as operational risk and trading risk. In those cases, the broad correlation specification approach in top-down approaches seems easier than specifying the full correlation matrix between all risk drivers for trading risk and operational risk.

Current risk aggregation models in banks range from very simple models that add sub-risks together to linear risk aggregation, and in some cases, risk aggregation using copula models. Also, some banks may use a combination of bottom-up and top-down approaches to risk integration.
Recently Basel (2008) addressed risk aggregation as one of the more challenging aspects of banks’ economic capital models. Recognizing that there has been some evolution in banks’ practices in risk aggregation, Basel noted that banks’ approaches to risk aggregation are generally less sophisticated than sub-risk measurement methodology. One of the main concerns in risk aggregation and calculation of economic capital is their calibration and validation – requiring a substantial amount of historical time-series data. Most banks do not have the necessary data available and may have to rely partially on expert judgment in calibration.

Following the importance of risk aggregation in banks’ economic capital models, there is vast literature on top-down risk aggregation. For example, Kuritzkes, Schuermann and Weiner (2002) consider a linear risk aggregation in financial conglomerates such as bancassurance. Rosenberg and Schuermann (2004) study copula-based aggregation of banks’ market, credit and operational risks using a comparison of the t- and normal copula. See also Dimakos and Aas (2004) and Cech (2006) for comparison of the properties of different copula models in top-down risk aggregation.

Risk Aggregation Methodologies

Risk aggregation involves the aggregation of individual risk measurements using a model for aggregation. The model for aggregation can be based on a simple linear aggregation or using a copula model. The linear aggregation model is based on aggregating risk, such as value at risk (VaR) or expected shortfall (ES), using correlations and the individual VaR or ES risk measures. The copula model aggregates risk using a copula for the co-dependence, such as the normal or t-copula, and the individual risk’s profit and loss simulations. The copula model allows greater flexibility in defining the dependence model than the linear risk aggregation. Below we review both of these approaches to risk aggregation.

Linear Risk Aggregation

The linear model for risk aggregation takes the individual VaR or ES risks as inputs and aggregates the risks using the standard formula for covariance. That is,

$$\text{VaR}(\alpha) = \sqrt{\sum_{i=1}^{n} \text{VaR}_i^2(\alpha) + 2 \sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{ij} \text{VaR}_i(\alpha) \text{VaR}_j(\alpha)}$$
where $\rho_{ij}$ is the correlation between risks $i$ and $j$ and $\alpha$ is the confidence level of the aggregate VaR. In this model there are two extreme cases. That is, the zero-correlation aggregate VaR,

$$\text{VaR}(\alpha) = \sqrt{\sum_{i=1}^{n} \text{VaR}_i^2(\alpha)}$$

and the maximum correlation VaR ($\rho_{ij} = 1$ for all $i, j$)

$$\text{VaR}(\alpha) = \sum_{i=1}^{n} \text{VaR}_i(\alpha).$$

The maximum correlation model is referred to as the simple summation model in Basel (2008). More generally, the linear risk aggregation can be performed for economic capital measure, EC, such that for EC risks $EC_1, \ldots, EC_n$ we have

$$\text{EC}(\alpha) = \sqrt{\left(\begin{array}{c} EC_1(\alpha) \\ \vdots \\ EC_n(\alpha) \end{array}\right)'} \left(\begin{array}{cccc} 1 & \rho_{12} & \cdots & \rho_{1n} \\ \rho_{21} & 1 & \cdots & \rho_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{n1} & \rho_{n2} & \cdots & 1 \end{array}\right) \left(\begin{array}{c} EC_1(\alpha) \\ \vdots \\ EC_n(\alpha) \end{array}\right).$$

Table 1 displays an example risk aggregation using the linear risk aggregation model. The sub-risks are market risk, credit risk, operational risk, funding risk and other risks (e.g., business risks). In the correlated risk aggregation, we have used the correlation matrix

$$\Sigma = \begin{pmatrix} 1 & 0 & 0.25 & 0.25 & 0.75 \\ 0 & 1 & 0.25 & 0.25 & 0.5 \\ 0.25 & 0.25 & 1 & 0.25 & 0.25 \\ 0.75 & 0.5 & 0.25 & 1 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 & 1 \end{pmatrix}$$

between the risks. The independent aggregate risk is obtained using zero correlations and the additive risk is obtained using correlations equal to unity.

Finally, we also calculate the actual contribution risk as a percentage of total risk. The contribution is the actual risk contribution from the sub-risks obtained in the context of the portfolio of total risks. This risk contribution is often compared with standalone risk to obtain a measure of the diversification level. For example, the market risk sub-risk has a standalone risk share of 16.3 percent whereas the risk contribution share is 11.89 percent. This represents a diversification level of about 73 percent compared to the simple summation approach.

**Table 1: Risk aggregation using the linear risk aggregation model.**

<table>
<thead>
<tr>
<th>Sub-Risks</th>
<th>Aggregated Risk</th>
<th>Independent Risk</th>
<th>Additive Risk</th>
<th>Contribution Risk (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total Risk</td>
<td>1524.85</td>
<td>1125.70</td>
<td>2013.00</td>
<td>100,0</td>
</tr>
<tr>
<td>Market Risk</td>
<td>250</td>
<td>250</td>
<td>250</td>
<td>11,89</td>
</tr>
<tr>
<td>Credit Risk</td>
<td>312</td>
<td>312</td>
<td>312</td>
<td>12,37</td>
</tr>
<tr>
<td>Operational Risk</td>
<td>119</td>
<td>119</td>
<td>119</td>
<td>3,03</td>
</tr>
</tbody>
</table>

1 In the section on capital allocation, we define the risk contribution as the derivative of total risk with respect to the sub-risks. That is, the derivative of $\text{VaR}(\alpha)$ with respect to risk exposure $i$. 
The linear model of risk aggregation is a very convenient model to work with. The only data required is the estimates of the sub-risks’ economic capital and the correlation between the sub-risks. However, the model also has some serious drawbacks. For example, the model has the assumption that quantiles of portfolio returns are the same as quantiles of the individual return – a condition that is satisfied in case the total risks and the sub-risks come from the same elliptic density family\(^2\).

The linear risk aggregation model is the aggregation model used in the new solvency regulations for the standard approach.

**Copula-Based Risk Aggregation**

While linear risk aggregation only requires measurement of the sub-risks, copula methods of aggregation depend on the whole distribution of the sub-risks. One of the main benefits of copula is that it allows the original shape of the sub-risk distributions to be retained. Further, the copula also allows for the specification of more general dependence models than the normal dependence model.

**Overview of Copulas**

To get a rough classification of available copulas, divide them into elliptic copulas (which are based on elliptic distributions), non-elliptic parametric copulas and copulas consistent with the multivariate extreme value theory. There are also other constructions based on transformations of copulas, including copulas constructed by using Bernstein polynomials. (See Embrechts et al. (2001) and Nelsen (1999) for an overview of different copulas.)

Implicit in the use of copulas as a model of co-dependence is the separation of the univariate marginal distributions modeling and the dependence structure. In this setting, the model specification framework consists of two components that can be constructed, analyzed and discussed independently. This gives a clear, clean, flexible and transparent structure to the modeling process.

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\(^2\) See Rosenberg and Schuermann (2004) for a derivation of the linear aggregation model and its special case of aggregating the multivariate normal distribution.
When introducing copulas, we start with the distribution function of the variables, $X_1, X_2, ..., X_n$.

$$F(x_1, x_2, ..., x_n) = P(X_1 \leq x_1, ..., X_n \leq x_n)$$

The copula is then simply the distribution of the uniforms. That is,

$$F(x_1, x_2, ..., x_n) = P(F_1(x_1) \leq F_1(x_1), ..., F_n(x_n) \leq F_n(x_n))$$

$$= P(U_1(x_1) \leq F_1(x_1), ..., U_n(x_n) \leq F_n(x_n)).$$

For example, this means that a two-dimensional copula, $C$, is a distribution function on the unit square $[0,1] \times [0,1]$ with uniform marginals.

Figure 1 displays three different two-dimensional copulas: the perfect negative dependence copula (left panel); the independence copula (middle panel); and the perfect dependence copula (right panel). The perfect negative dependence copula has no mass outside the northeast corner, whereas the perfect dependence copula has no mass outside the diagonal. In contrast, the independence copula has mass almost everywhere.

![Figure 1: Some special copulas: the perfect negative dependence (left panel); the independence copula (middle panel); and the perfect dependence copula (right panel).](image)

Since the copula is defined as the distribution of the uniform margins, the copula for a multivariate density function can be obtained using the method of inversion. That is,

$$C(u, v) = F_{x,y}^{-1}(F_x^{-1}(u), F_y^{-1}(v))$$

where $C(u,v)$ is the copula, $u$ and $v$ are uniforms, $F_{x,y}$ is the joint cumulative density function, and $F_x^{-1}$ and $F_y^{-1}$ are the inverse marginal quantile functions. In the special case that the joint and the marginal densities and copula are normal, the relationship can be written as:

$$C(u, v) = \Phi(\phi^{-1}(u), \phi^{-1}(v))$$

where $\Phi$ is the normal cumulative density and $\phi$ is the normal marginal density.
Figure 2 displays different elliptic and Archimedean copula families and some of their members. For example, the elliptic family includes the well-known normal and t-copulas, and the Archimedean class of copulas includes the Gumbel, Clayton and Frank copulas.

In the elliptic class of copulas, the normal copula is the copula of the normal distribution and the student-t copula is the copula of the student t-distribution. The Archimedean class of copulas displays three well-known members of this class. These are the Clayton, Gumbel and Frank copula. All of these Archimedean copulas have an explicit copula representation, which makes them easy to use for simulation purposes. The Clayton two-dimensional copula is represented as

\[ C(u, v) = (u^{-\theta} + v^{-\theta} - 1)^{-1/\theta} \]

and as \( \theta \) approaches 0 we obtain the independence copula, whereas as \( \theta \) approaches \( \infty \) we obtain the perfect dependence copula. The Gumbel copula has two-dimensional representation

\[ C(u, v) = \exp \left\{ -((-\log u)^\theta + (-\log v)^\theta - 1)^{1/\theta} \right\} \]

where \( \theta = 1 \) implies the independence copula and as \( \theta \) approaches \( \infty \) we obtain the perfect dependence copula. Finally, the Frank copula has the bivariate representation

\[ C(u, v) = -\frac{1}{\theta} \log \left\{ 1 + \frac{\exp(-\theta u) - 1}{\exp(-\theta) - 1} \right\} \]

The Frank copula displays perfect negative dependence for \( \theta \) equal to \(-\infty\) and perfect positive dependence for \( \theta \) equal to \(\infty\).

**Measuring Copula Dependence**

The most well-known copula dependence measures are the Spearman’s rho, the Kendall’s tau and the coefficients of upper and lower tail dependence (Embrechts, et al. (1999)). These are all bivariate notions that can be extended to the multivariate case by applying them to all pairs of components in the vector. The linear correlation, \( \rho \), between \( X_1, X_2 \) is

\[ \rho(X_1, X_2) = \frac{\text{Cov}(X_1, X_2)}{\sqrt{\text{Var}(X_1)\text{Var}(X_2)}} \]
Linear correlation is, as the word indicates, a linear measure of dependence. For constants \( \alpha, \beta, c_1, c_2 \) it has the property that

\[
\rho(\alpha X_1 + c_1, \beta X_2 + c_2) = \rho(X_1, X_2)
\]

and hence is invariant under strictly increasing linear transformations. However, in general

\[
\rho(X_1, X_2) \neq \rho(F_1(X_1), F_2(X_2))
\]

for strictly increasing transformations, \( F_1 \) and \( F_2 \) and hence \( \rho \) is not a copula property as it depends on the marginal distributions.

For two independent vectors of random variables with identical distribution function, \((X_1, X_2)\) and \((X^*, X^*)\), Kendall’s tau is the probability of concordance minus the probability of discordance, i.e.,

\[
\tau(X_1, X_2) = P[(X_1 - X^*)_1(X_2 - X^*)_2] > 0] - P[(X_1 - X^*)_1(X_2 - X^*)_2] < 0]
\]

Kendall’s tau, \( \tau \), can be written as

\[
\tau(X_1, X_2) = 4 \int_0^1 \int_0^1 C(u, v) dC(u, v) - 1
\]

and hence is a copula property. Spearman’s rho is the linear correlation of the uniform variables \( u = F_1(X_1) \) and \( v = F_1(X_2) \) i.e.,

\[
\rho_S(X_1, X_2) = \rho(F_1(X_1), F_2(X_2))
\]

and the Spearman’s rho, \( \rho_S \), is also a property of only the copula \( C \) and not the marginals. Often the Spearman’s rho is referred to as the correlation of ranks. We note that Spearman’s rho is simply the usual linear correlation, \( \rho \), of the probability-transformed random variables.

Another important copula quantity is the coefficient of upper and lower tail dependence. The coefficient of upper tail dependence \( \Lambda_U(X_1, X_2) \) is defined by

\[
\Lambda_U(X_1, X_2) = \lim_{u, v \to 1} P[X_1 > F_1^{-1}(u) | X_2 > F_2^{-1}(v)]
\]

provided that the limit \( \Lambda_U(X_1, X_2) \in [0, 1] \) exists. The coefficient of lower tail dependence, \( \Lambda_L(X_1, X_2) \), is similarly defined by

\[
\Lambda_L(X_1, X_2) = \lim_{u, v \to 0} P[X_1 > F_1^{-1}(u) | X_2 > F_2^{-1}(v)]
\]
provided that the limit $\Lambda_L(X_1, X_2) \in [0, 1]$ exists. If $\Lambda_U(X_1, X_2) \in (0, 1]$, then $X_1$ and $X_2$ are said to have asymptotic upper tail dependence. If $\Lambda_U(X_1, X_2) = 0$, then $X_1$ and $X_2$ are said to have asymptotic upper tail independence. The obvious interpretation of the coefficients of upper and lower tail dependence is that these numbers measure the probability of joint extremes of the copula.

**The Normal and t Copulas**

The Gaussian copula is the copula of the multivariate normal distribution. The random vector $X = (X_1, \ldots, X_1)$ is multivariate normal if and only if the univariate margins $F_1, \ldots, F_1$ are Gaussians and the dependence structure is described by a unique copula function $C$, the normal copula, such that

$$C_N(u_1, \ldots, u_n) = \Phi(\phi^{-1}(u_1), \ldots, \phi^{-1}(u_n))$$

where $\Phi$ is the standard multivariate normal distribution function with linear correlation matrix $\Sigma$ and $\phi^{-1}$ is the inverse of standard univariate Gaussian distribution function. For the Gaussian copula there is an explicit relation between the Kendall's tau, $\tau$, the Spearman's rho, $\rho_S$, and the linear correlation, $\rho$, of the random variables $X_1, X_2$. In particular,

$$\tau(X_1, X_2) = \frac{2}{n} \arcsin(\rho)$$

$$\rho_S(X_1, X_2) = \frac{6}{n} \arcsin(0.5\rho)$$

The copula of the multivariate student t distribution is the student t copula. Let $X$ be a vector with an $n$-variate student t distribution with $\nu$ degrees of freedom, mean vector $\mu$ (for $\nu > 1$) and covariance matrix $(\nu/(\nu-2))\Sigma$ (for $\nu > 2$). It can be represented in the following way:

$$X \sim \mu + \frac{\sqrt{\nu} Z}{\sqrt{\Sigma}}$$

where $\mu \in \mathbb{R}^n$, $S \sim \chi^2(\nu)$ and the random vector $Z \sim \mathcal{N}(0, \Sigma)$ is independent of $S$. The copula of the vector $X$ is the student t copula with $\nu$ degrees of freedom. It can be analytically represented as

$$C_t(u_1, \ldots, u_n) = t_n(t_{\nu}^{-1}(u_1), \ldots, t_{\nu}^{-1}(u_n))$$

where $t_n$ denotes the multivariate distribution function of the random vector $\frac{\sqrt{\nu} Z}{\sqrt{\Sigma}}$ and $t_{\nu}^{-1}$ denotes the inverse margins of $t_n$. For the bivariate t copula, there is an explicit relation between the Kendall's tau and the linear correlation of the random variables $X_1, X_2$. Specifically,

$$\tau(X_1, X_2) = \frac{2}{n} \arcsin(\rho)$$
However, the simple relationship between the Spearman’s rho, $\rho_S$, and the linear correlation, $\rho$, that we had for the Gaussian copula above does not exist in the t-copula case.

To estimate the parameters of the normal and t-copulas there is, due to the explicit relationships between Kendall’s tau and linear correlation, a simple method based on Kendall’s tau. The method consists of constructing an empirical estimate of linear correlation for each bivariate margin and then using the relationship between the linear correlation and Kendall’s tau stated above to infer an estimate. An alternative method employs the linear correlation of the probability-transformed random variables (i.e., the Spearman’s rho, or in the case of the Gaussian, the explicit relation between linear correlation and Spearman’s rho).

Figure 3 displays the (for $u \leq 1$) tail dependence of the bivariate normal and t-copula with 1 degree of freedom for different values of the correlation parameter, $\rho$, and different values for $u$. We note from the table that the decay of tail dependence is quite fast for the normal copula. Indeed, asymptotically the normal copula has no tail dependence (except for the limiting case of $\rho=1$) and this has led many researchers to question the use of the normal copula. In practice, real differences between the copulas are expected only for high quantiles.

Figure 3: Tail dependence for the normal copula (left panel) and t-copula with 1 degree of freedom (right panel) for different correlations, $\rho$, and values of $u$. 
Normal Mixture Copulas

A normal mixture distribution is the distribution of the random vector

$$X = f(W) + W2$$

where $$f(W) \in \mathbb{R}^n$$, $$W \geq 0$$ is a random variable and the random vector $$Z \sim N(0, \Sigma)$$ is independent of $$W$$. In particular, if $$f(W) = \mu \in \mathbb{R}^n$$ and $$W$$ is distributed as an inverse gamma with parameters $$(1/2)v,(1/2)v$$, then $$X$$ is distributed as a t-distribution.

Another type of normal mixture distribution is the discrete normal mixture distribution, where $$f(W) = \mu$$ and $$W$$ is a discrete random variable taking values $$w_1, w_2$$ with probability $$p_1, p_2$$. By setting $$w_2$$ large relative to $$w_1$$ and $$p_1$$ large relative to $$p_2$$, one can interpret this as two states – one ordinary state and one stress state. The normal mixture distribution where $$f(W) = \mu + W \gamma$$ and $$\gamma$$ is different among at least one of the components $$\gamma_1, \ldots, \gamma_n$$ is a non-exchangeable normal mixture distribution, and $$\gamma$$ is called an asymmetry parameter. Negative values of $$\gamma$$ produce a greater level of tail dependence for joint negative returns. This is the case that is perceived as relevant for many financial time series where joint negative returns show stronger dependence than joint positive returns. The copula of a normal mixture distribution is a normal mixture copula.

Aggregate Risk Using Copulas

When aggregating risk using copulas, the full profit and loss density of the sub-risks, as well as the choice of copula and copula parameter(s), is required. This is only marginally more information than is required for the linear aggregation model. In particular, for the linear risk aggregation model we only needed the risk measures of the sub-risks. Using a copula model for aggregation, additional correlation parameters may be required (e.g., for the student t-copula we also require a degree of freedom parameter).
In Table 2 below, we calculate value at risk (VaR) and expected shortfall (ES) copula aggregations using the normal, t-copula and normal mixture copula for a 99 percent confidence level. The risk aggregation using the normal mixture copula is evaluated using a bivariate mixture with parameterization NMIX(p1,p2,x1,x2). Here, p1 and p2 are the probabilities of normal states 1 and 2 respectively and x1, x2 are the mixing coefficients. The individual risk’s profit and loss distributions are simulated from standard normal densities, and the correlated aggregation correlation matrix is the same as the one used for Table 1 above. Table 2 shows that the copula risk aggregation VaR and ES increase when a non-normal copula model, such as the t-copula or the normal mixture copula, is used for aggregation. For the t-copula, the aggregate risk generally increases the lower the degrees of freedom parameter, and for the normal mixture copula, the aggregate risk generally increases with the mixing coefficient. For a more detailed analysis of these copula models in the context of bottom-up risk aggregation for market risk, we refer to Skoglund, Erdman and Chen (2010).

Table 2: Copula value at risk and expected shortfall risk aggregation using the normal copula, t-copula and the normal mixture copula.

<table>
<thead>
<tr>
<th>Copula Model</th>
<th>Aggregate VaR (99%)</th>
<th>Aggregate ES (99%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Normal</td>
<td>7.74</td>
<td>8.70</td>
</tr>
<tr>
<td>T(10)</td>
<td>7.93</td>
<td>9.28</td>
</tr>
<tr>
<td>T(5)</td>
<td>8.16</td>
<td>9.67</td>
</tr>
<tr>
<td>NMIX(0,9,0.1,1,10)</td>
<td>8.06</td>
<td>9.94</td>
</tr>
<tr>
<td>NMIX(0,9,0.1,1,400)</td>
<td>8.91</td>
<td>10.82</td>
</tr>
</tbody>
</table>

Capital Allocation

Having calculated aggregate risk using a method of risk aggregation, the next step is to allocate risks to the different sub-risks. In particular, banks allocate risk/capital for the purpose of:

- Understanding the risk profile of the portfolio.
- Identifying concentrations.
- Assessing the efficiency of hedges.
- Performing risk budgeting.
- Allowing portfolio managers to optimize portfolios based on risk-adjusted performance measures.

The fact that the individual profit and loss distributions are simulated from the standard normal distribution is without loss of generality. In principle, the underlying profit and loss distributions can be simulated from any density.
In practice, capital allocation is based on risk contributions to risk measures such as aggregate economic capital. In order to achieve consistency in risk contributions when economic capital model risks differ from actual capital, banks regularly scale economic capital risk contributions to ensure risk contributions equal the sum of total capital held by the bank.

**Defining Risk Contributions**

Following Litterman (1996) – introducing the risk contributions in the context of the linear normal model – there has been a focus on risk contribution development for general non-linear models. Starting with Hallerbach (1998), Tasche (1999) and Gourieroux et al. (2000), general value at risk and expected shortfall contributions were introduced. These contributions utilize the concept of the Euler allocation principle. This is the principle that a function, $\psi$, that is homogenous of degree 1 and continuously differentiable satisfies

$$
\psi = \sum_{i=1}^{n} \frac{\partial \psi}{\partial w_i}
$$

That is, the sum of the derivative of the components of the function, $\psi$, sum to the total. If we interpret $\psi$ as a risk measure and $w_i$ as the weights on the sub-risk’s, then the Euler allocation suggests that risk contributions are computed as first-order derivatives of the risk with respect to the sub-risk’s.

The Euler allocation is a theoretically appealing method of calculating risk contributions. In particular, Tasche (1999) shows that this is the only method that is consistent with local portfolio optimization. Moreover, Denault (2001) shows that the Euler decomposition can be justified economically as the only “fair” allocation in co-operative game theory (see also Kalkbrener (2005) for an axiomatic approach to capital allocation and risk contributions). These properties of Euler risk contributions have led to the Euler allocation as the standard metric for risk; it’s also used by portfolio managers for risk decomposition into subcomponents such as instruments or sub-risks. Recently, Tasche (2007) summarized the to-date findings and best-practice usage of risk contributions based on Euler allocations; Skoglund and Chen (2009) discussed the relation between Euler risk contributions and so-called risk factor information measures.

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4 Risk factor information measures attempt to break down risk into the underlying risk factors such as equity, foreign exchange and commodity. The risk factor information measure is in general not a decomposition in the sense of risk contributions.
In the copula risk aggregation, the input for the sub-risks is a discrete simulation of profits and losses. Hence, strictly, the derivative of aggregate risk with respect to the sub-risks does not exist. How do we define risk decomposition in this case? The answer lies in the interpretation of risk contributions as conditional expectations (see Tasche (1999), Gourieroux et al. (2000)). This means that we can estimate VaR contributions for the copula risk aggregation technique using the conditional expectation
\[
\frac{d\text{VaR}(\alpha)}{d\omega_i} = E(L_i | L = \text{VaR}(\alpha))
\]
where this is the risk contribution for sub-risk i, L is the aggregate loss and L_i is the loss of sub-risk i. The expected shortfall (ES) risk measure is defined as the average loss beyond VaR. That is,
\[
\text{ES}(\alpha) = \frac{1}{1 - \alpha} \int_{\alpha}^{1} \text{VaR}(u)\,du
\]
Hence, taking the expectation inside the integral, ES risk contributions can be calculated as conditional expectations of losses beyond VaR.

Table 3 below illustrates the computation of discrete risk contributions using an example loss matrix. In particular if the tail scenario 5 corresponds to the aggregate VaR, that is 39,676, then the VaR contributions for the sub-risks correspond to the realized loss in the sub-risks corresponding VaR scenario. Specifically, the VaR contribution for sub-risk 1 is then -0,086, the VaR contribution for sub-risk 2 is 44,042 and for sub-risk 3 is -1,975. By definition these contributions sum to aggregate VaR. The ES is defined as the expected loss beyond VaR and is hence the average of aggregate losses beyond VaR. This average is, from Table 3, 90,9. The ES contributions are respectively for sub-risk 1, 2 and 3: 0,1107, 90,74, and 0,0945. These contributions are obtained by averaging the sub-risks’ corresponding ES scenarios for tail scenario 1, 2, 3 and 4. Again, as in the VaR case, the sum of ES contributions equals the aggregate ES.

Table 3: Example loss scenarios for aggregate risk and sub-risks 1, 2 and 3, and the computation of VaR and ES risk contributions.
Having calculated risk contributions, the next step is to allocate capital. Since actual capital may not be equal to aggregated risk, a scaling method is used. That is, the capital contribution is obtained by scaling the risk contribution by the capital and aggregate risk ratio.

**Issues in Practical Estimation of Risk Contributions**

When estimating risk contributions, it is important that the risk contributions are stable with respect to small changes in risk; otherwise, the information about the sub-risks contribution to aggregate risk will not be meaningful. Indeed, if small changes in risk yield large changes in risk contributions then the amount of information about the risk contained in the risk contributions is very small. More formally, we may express this requirement of risk smoothness by saying that the risk measure must be convex. Artzner et al. (1999) introduces the concept of coherent risk measures. A coherent risk measure, \( m \), has the property that:

\[
\begin{align*}
m(X+Y) &\leq m(X) + m(Y) & \text{(subadditive)} \\
m(tX) &= tm(X) & \text{(homogenous of degree one)} \\
\text{If } X > Y \text{ a.e. then } m(X) &> m(Y) & \text{(monotonous)} \\
m(X+a) &= m(X) + a & \text{(risk-free condition).}
\end{align*}
\]

The first and the second condition together imply that the risk measure is convex. Unfortunately VaR is not a convex risk measure in general; in particular, this implies that the VaR contributions may be non-smooth and erratic.

Figure 5 displays the VaR measure (y-axis) when the position (x-axis) is changed. Notice that the VaR curve is non-convex and non-smooth. Considering the calculations of VaR contributions in this case may be misleading, as the VaR contributions change wildly even for small changes in underlying risk position.

![Figure 5: VaR is in general a non-convex risk measure such that the risk profile is non-convex (i.e., non-smooth).](image)
The above properties of VaR contributions have led many practitioners to work with expected shortfall, and in particular, risk contributions to expected shortfall. In contrast to VaR, expected shortfall is a coherent risk measure and hence can be expected to have smooth risk contributions. Alternatively, VaR is used as the risk measure while risk contributions are calculated using expected shortfall risk and subsequently scaled down to match aggregate VaR. Other approaches to smoothing VaR contributions include averaging the losses close to the VaR point, or smoothing the empirical measure by Kernel estimation (Epperlein and Smillie (2006)).

Risk contributions are also key in a bank’s approach to measuring and managing risk concentrations. In particular, a bank’s framework for concentration risk and limits should be well-defined. In Tasche (2006), Memmel and Wehn (2006); Garcia, Cespedes et al. (2006); and Tasche (2007), the ratio of Euler risk contribution to the standalone risk is defined as the diversification index. For coherent risk measures, the diversification index measures the degree of concentration such that a diversification index of 100 percent for a sub-risk may be deemed to have a high risk concentration. In contrast, a low diversification index, say 60 percent, may be considered to be a well-diversified sub-risk. Note that the diversification index measures the degree of diversification of a sub-risk in the context of a given portfolio. That is, the risk, as seen from the perspective of a given portfolio, may or may not be diversified.

Value-Based Management Using Economic Capital

The economic capital and the allocation of economic capital is used in a bank’s performance measurement and management to distribute the right incentives for risk-based pricing, ensuring that only exposures that contribute to the bank’s performance targets are considered for inclusion in the portfolio. In principle, banks add an exposure to a portfolio if the new risk-adjusted return is greater than the existing, i.e.

\[ \frac{r_{n+1}}{RC_{n+1}} > RAROC(w) \]

Here, \( RC_{n+1} \) is the risk contribution of the new exposure with respect to economic capital and \( r_{n+1} \) is the expected return of the new exposure. Value is created, relative to the existing portfolio, if the above condition is satisfied. For further discussion on risk-adjusted performance management for traditional banking book items, see the SAS white paper *Funds Transfer Pricing and Risk-Adjusted Performance Measurement*. 
Conclusion

Risk aggregation and estimation of overall risk is key in banks’ approaches to economic capital and capital allocation. The resulting capital also forms the basis for banks’ value-based management of the balance sheet.

When estimating aggregate capital, one typically uses a combination of bottom-up and top-down risk aggregation approaches. The top-down approach to risk capital involves the selection of the aggregation model, such as the linear aggregation or copula aggregation model, and the estimation of the dependence between aggregate risks using historical and/or expertise data. For the linear risk aggregation model, the correlations between the risks and the individual risk levels are needed for risk aggregation. The copula aggregation model, being a more general approach to risk aggregation, uses information about the complete profit and loss density of the underlying risks and typically requires additional parameters to correlation to fully capture dependence between risks. In practice, both approaches have their own drawbacks and benefits. It is therefore advisable to consider multiple approaches to risk aggregation.

This SAS white paper has presented the linear and copula model approaches to the calculation of risk aggregations and economic capital. We have also discussed the use of risk contributions as a basis for allocating economic capital. Finally, we demonstrated how to integrate economic capital into risk adjusted performance.

About SAS® Risk Management for Banking

SAS Risk Management for Banking has been designed as a comprehensive and integrated suite of quantitative risk management applications, combining market risk, credit risk, asset and liability management, and firmwide risks into one solution. The solution leverages the underlying SAS 9.2 platform and the SAS Business Analytics Framework, providing users with a flexible and modular approach to risk management. Users can start with one of the predefined workflows and then customize or extend the functionality to meet ever-changing risk and business needs. The introduction of SAS Risk Management for Banking will take the industry’s standards to a higher paradigm of analytics, data integration and risk reporting.
References


